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Irreducible representations of $U_{p,q}[gl(2/2)]$

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Abstract
The two-parametric quantum superalgebra $U_{p,q}[gl(2/2)]$ and its representations are considered. All finite-dimensional irreducible representations of this quantum superalgebra can be constructed and classified into typical and non-typical ones according to a proposition proved in the present paper. This proposition is a non-trivial deformation from the one for the classical superalgebra $gl(2/2)$, unlike the case of one-parametric deformations.

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1. Introduction

The quantum groups [1–6] were introduced in the 80s as a result of the study on quantum integrable systems and Yang–Baxter equations (YBEs) [7]. At first sight, it turns out that they are related to unrelated areas of both physics and mathematics and, therefore, have been intensively investigated in various aspects, including their applications (see references [1–15] and references therein). For applications of quantum groups, as in the non-deformed cases, we often need their explicit representations, in particular, the finite dimensional ones which in many cases are connected with rational and trigonometric solutions of the quantum YBEs [1–8]. However, in spite of the effort and remarkable results in this direction, the problem of investigating and constructing explicit representations of quantum groups, especially those for quantum superalgebras, is still far from being satisfactorily solved. Even in the case of one-parametric quantum superalgebras, explicit representations are mainly known for quantum Lie superalgebras of lower ranks and of particular types like $U_q[osp(1/2)]$ and $U_q[gl(1/n)]$ (references [15–17]). So far, finite-dimensional representations of some bigger quantum superalgebras such as $U_q[osp(1/2n)]$ and $U_q[gl(m/n)]$ with $m, n > 2$ have been considered but have not been explicitly constructed (see, for example, [18, 19]). At the moment, detailed results in this aspect are known only for the cases with both $m, n \leqslant 2$ considered in [15, 20, 21], while for $U_q[gl(m/n)]$ with arbitrary m and n not all finite-dimensional representations but only a, although big, class of representations called essentially typical is known [22].

As far as the multi-parametric deformations (first considered in [3]) are concerned, this area is even less covered and results are much poorer. Some types of two-parametric deformations have been considered by a number of authors from different points of view (see [23, 24] and references therein) but, to our knowledge, explicit representations are known and/or classified in a few lower rank cases such as $U_{p,q}[sl(2/1)]$ and $U_{p,q}[gl(2/1)]$ only [23, 25]. The latter two-parametric quantum superalgebra $U_{p,q}[gl(2/1)]$ was consistently introduced and investigated in [23] where all its finite-dimensional irreducible representations were explicitly constructed and classified at generic deformation parameters. This $U_{p,q}[gl(2/1)]$, however, is still a small quantum superalgebra which can be defined without the so-called extra-Serre defining relations [26–28] representing additional constraints on odd Chevalley generators in higher rank cases. In order to include the extra-Serre relations on examination we introduced and considered a bigger two-parametric quantum superalgebra, namely $U_{p,q}[gl(2/2)]$ and its representations [24, 29]. Our other motivation for considering this quantum superalgebra is that already in the non-deformed case, the superalgebras $gl(n/n)$, especially their subalgebras $sl(n/n)$ and $psl(n/n)$, have special properties (in comparison with other $gl(m/n)$, $m \neq n$) and, therefore, attract interest [30–32]. Additionally, structures of two-parameter deformations investigated in [23, 24, 29] and here are, of course, richer than those of one-parameter deformations. Every deformation parameter can be independently chosen to take a separate generic value (including zero) or to be a root of unity.

Combining the advantages of the previously developed methods [20, 21, 23] for $U_q[gl(2/2)]$ and $U_{p,q}[gl(1/2)]$, we described in [24] how to construct finite-dimensional representations of the two-parametric quantum Lie superalgebra $U_{p,q}[gl(2/2)]$. In this paper we consider when these representations constructed are irreducible. It turns out that they can be classified again into typical and non-typical representations which, even at generic deformation parameters, however, are non-trivial deformations from the classical analogues [33], unlike many cases of one-parametric deformations.

2. The quantum superalgebra $U_{p,q}[gl(2/2)]$

The quantum superalgebra $U_{p,q} \equiv U_{p,q}[gl(2/2)]$, as a two-parametric deformation of the universal enveloping algebra $U[gl(2/2)]$ of the Lie superalgebra $gl(2/2)$, can be completely generated by the operators L_k, E_{12}, E_{23}, E_{34}, E_{21}, E_{32}, E_{43} and E_{ii} ($1 \leqslant i \leqslant 4$) again called Cartan–Chevalley generators subject to the following (defining) relations [24, 29]:

(a) *Super-commutation relations* ($1 \leqslant i, i+1, j, j+1 \leqslant 4$):

$$[E_{ii}, E_{jj}] = 0 \tag{1a}$$

$$[E_{ii}, E_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1})E_{j,j+1} \tag{1b}$$

$$[E_{ii}, E_{j+1,j}] = (\delta_{i,j+1} - \delta_{ij})E_{j+1,j} \tag{1c}$$

$$[\text{even generator}, L_k] = 0, k = 1, 2, 3 \tag{1d}$$

$$[E_{i,i+1}, E_{j+1,j}\} = \delta_{ij}\left(\frac{q}{p}\right)^{L_i - H_i(1+\delta_{i2})/2}[H_i] \tag{1e}$$

(b) *Serre relations*:

$$[E_{12}, E_{34}] = [E_{21}, E_{43}] = 0 \tag{2a}$$

$$E_{23}^2 = E_{32}^2 = 0 \tag{2b}$$

$$[E_{12}, E_{13}]_p = [E_{21}, E_{31}]_q = [E_{24}, E_{34}]_q = [E_{42}, E_{43}]_p = 0 \tag{2c}$$

(c) *Extra-Serre relations*:

$$\{E_{13}, E_{24}\} = 0 \tag{3a}$$

$$\{E_{31}, E_{42}\} = 0 \tag{3b}$$

where $H_i \equiv (E_{ii} - \frac{d_{i+1}}{d_i} E_{i+1,i+1})$, $d_1 = d_2 = -d_3 = -d_4 = 1$, $L_1 \equiv L_l$, $L_2 \equiv 0$, $L_3 \equiv L_r$ (with L_l and L_r explained later), $[x] \equiv (q^x - p^{-x})/(q - p^{-1})$ is a so-called pq-deformation of x being a number or an operator and, finally, [, } is a notation for the supercommutators. Here, the operators

$$E_{13} := [E_{12}, E_{23}]_{q^{-1}} \tag{4a}$$

$$E_{24} := [E_{23}, E_{34}]_{p^{-1}} \tag{4b}$$

$$E_{31} := -[E_{21}, E_{32}]_{p^{-1}} \tag{4c}$$

$$E_{42} := -[E_{32}, E_{43}]_{q^{-1}} \tag{4d}$$

and the operators composed in the following way:

$$E_{14} := [E_{12}, [E_{23}, E_{34}]_{p^{-1}}]_{q^{-1}} \equiv [E_{12}, E_{24}]_{q^{-1}} \tag{5a}$$

$$E_{41} := [E_{21}, [E_{32}, E_{43}]_{q^{-1}}]_{p^{-1}} \equiv -[E_{21}, E_{42}]_{p^{-1}} \tag{5b}$$

are defined as new generators, where $[A, B]_r = AB - rBA$. These generators, like E_{23} and E_{32}, are all odd and have vanishing squares. The generators E_{ij}, $1 \leqslant i, j \leqslant 4$, are two-parametric deformation analogues (pq-analogues) of the Weyl generators e_{ij} of the superalgebra $gl(2/2)$ whose universal enveloping algebra $U[gl(2/2)]$ is a classical limit of $U_{p,q}[gl(2/2)]$ when $p, q \to 1$. The so-called maximal-spin operator L_l (or L_r) is a constant within a finite-dimensional irreducible module (*fidirmod*) of a $U_{p,q}[gl(2)]$ (defined below) and are different for different $U_{p,q}[gl(2)]$-fidirmods. Therefore, commutators between these operators with the odd generators intertwining $U_{p,q}[gl(2)]$-fidirmods take concrete forms on concrete basis vectors. Other commutation relations between E_{ij} follow from the relations (1)–(3) and the definitions (4) and (5).

3. Representations of $U_{p,q}[gl(2/2)]$

The subalgebra $U_{p,q}[gl(2/2)_0]$ ($\subset U_{p,q}[gl(2/2)]_0 \subset U_{p,q}[gl(2/2)]$) is even and isomorphic to $U_{p,q}[gl(2) \oplus gl(2)] \equiv U_{p,q}[gl(2)] \oplus U_{p,q}[gl(2)]$, which can be completely generated by L_1, L_3, E_{12}, E_{34}, E_{21}, E_{43} and E_{ii}, $1 \leqslant i \leqslant 4$,

$$U_{pq}[gl(2/2)_0] = \text{lin.env.}\{L_1, L_3, E_{ij} \,\|\, i, j = 1, 2 \text{ and } i, j = 3, 4\}. \tag{6}$$

In order to distinguish two components $U_{p,q}[gl(2)]$ of $U_{p,q}[gl(2/2)_0]$ we set

$$\text{left } U_{p,q}[gl(2)] \equiv U_{p,q}[gl(2)_l] := \text{lin.env.}\{L_1, E_{ij} \,\|\, i, j = 1, 2\} \tag{7}$$

$$\text{right } U_{p,q}[gl(2)] \equiv U_{p,q}[gl(2)_r] := \text{lin.env.}\{L_3, E_{ij} \,\|\, i, j = 3, 4\} \tag{8}$$

that is

$$U_{p,q}[gl(2/2)_0] = U_{p,q}[gl(2)_l \oplus gl(2)_r]. \tag{9}$$

We see that each of the odd spaces A_+ and A_- spanned on the positive and negative odd roots (generators) E_{ij} and E_{ji}, $1 \leqslant i \leqslant 2 < j \leqslant 4$, respectively

$$A_+ = \text{lin.env.}\{E_{14}, E_{13}, E_{24}, E_{23}\} \tag{10}$$

$$A_- = \text{lin.env.}\{E_{41}, E_{31}, E_{42}, E_{32}\} \tag{11}$$

is a representation space of the even subalgebra $U_{p,q}[gl(2/2)_0]$ which, as seen from (1) and (2), is a stability subalgebra of $U_{p,q}[gl(2/2)]$. Therefore, we can construct representations of $U_{p,q}[gl(2/2)]$ induced from some (finite-dimensional irreducible, for example) representations of $U_{p,q}[gl(2/2)_0]$, which are realized in some representation spaces (modules) $V_0^{p,q}$ being tensor products of $U_{p,q}[gl(2)_l]$-modules $V_{0,l}^{p,q}$ and $U_{p,q}[gl(2)_r]$-modules $V_{0,r}^{p,q}$

$$V_0^{p,q}(\Lambda) = V_{0,l}^{p,q}(\Lambda_l) \otimes V_{0,r}^{p,q}(\Lambda_r) \tag{12}$$

where Λ's are some signatures (such as highest weights, respectively) characterizing the modules (highest weight modules, respectively). Here Λ_l and Λ_r are referred to as the left and the right components of Λ, respectively,

$$\Lambda = [\Lambda_l, \Lambda_r]. \tag{13}$$

If we demand

$$E_{23} V_0^{p,q}(\Lambda) = 0 \tag{14}$$

hence

$$U_{p,q}(A_+) V_0^{p,q} = 0 \tag{15}$$

we turn the $U_{p,q}[gl(2/2)_0]$-module $V_0^{p,q}$ into a $U_{p,q}(B)$-module where

$$B = A_+ \oplus gl(2) \oplus gl(2). \tag{16}$$

The $U_{p,q}[gl(2/2)]$-module $W^{p,q}$ induced from the $U_{p,q}[gl(2/2)_0]$-module $V_0^{p,q}$ is the factor space

$$W^{p,q} = W^{p,q}(\Lambda) = \left[U_{p,q} \otimes V_0^{p,q}(\Lambda)\right] / I^{p,q}(\Lambda) \tag{17}$$

which, of course, depends on Λ, where

$$U_{p,q} \equiv U_{p,q}[gl(2/2)] \tag{18}$$

while $I^{p,q}$ is the subspace

$$I^{p,q} = \text{lin.env.}\left\{ub \otimes v - u \otimes bv \| u \in U_{p,q}, b \in U_{p,q}(B) \subset U_{p,q}, v \in V_0^{p,q}\right\}. \tag{19}$$

Using the commutation relations (1)–(3) and the definitions (4) and (5) we can prove the analogue of the Poincaré–Birkhoff–Witt theorem. Consequently, a basis of $W^{p,q}$ can be constituted by taking all the vectors of the form

$$|\theta_1, \theta_2, \theta_3, \theta_4; (\lambda)\rangle := (E_{41})^{\theta_1}(E_{31})^{\theta_2}(E_{42})^{\theta_3}(E_{32})^{\theta_4} \otimes (\lambda) \qquad \theta_i = 0, 1 \tag{20}$$

where (λ) is a (Gel'fand–Zetlin, for example) basis of $V_0^{p,q} \equiv V_0^{p,q}(\Lambda)$. This basis of $W^{p,q}$ called the induced $U_{p,q}[gl(2/2)]$-basis (or simply, the induced basis), however, is not convenient for investigating the module structure of $W^{p,q}$. It was the reason the so-called reduced basis was introduced [24]. It is obvious that if the module $V_0^{p,q}$ is finite-dimensional so is the module $W^{p,q}$. In this case $W^{p,q}$ can be characterized by a signature $[m]$ and is decomposed into a direct sum of (16, at most) $U_{p,q}[gl(2/2)_0]$-fidirmod's $V_k^{p,q}$ of signatures $[m]_k$:

$$W^{p,q}([m]) = \bigoplus_{k=0}^{15} V_k^{p,q}([m]_k). \tag{21}$$

Thus, the reduced basis of $W^{p,q}$ is a union of the bases of all $V_k^{p,q}$'s which can be presented by the quasi-Gel'fand–Zetlin patterns [24], corresponding to the branching rule $U_{p,q}[gl(2/2)] \supset U_{p,q}[gl(2/2)_0] \supset U_{p,q}[gl(1) \otimes gl(1)]$,

$$\begin{bmatrix} m_{13} & m_{23} & m_{33} & m_{43} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{11} & 0 & m_{31} & 0 \end{bmatrix}_k \equiv (m)_k \qquad 0 \leqslant k \leqslant 15 \tag{22}$$

where m_{ij} are complex numbers such that $m_{i2} - m_{i1} \in \mathbf{Z}^+, m_{i1} - m_{i+1,2} \in \mathbf{Z}^+, m_{i3} - m_{i+1,3} \in \mathbf{Z}^+, i = 1, 3$. The second row $[m_{12}, m_{22}, m_{32}, m_{42}]$ in (22) is fixed for a given k, as for $k = 0$ it takes the value of the first row $[m_{13}, m_{23}, m_{33}, m_{43}]$ which is fixed for all $k = 0, 1, \ldots, 15$. Now, a signature $[m]_k$ of a $V_k^{p,q}$ is identified with a second row,

$$[m]_k \equiv [m_{12}, m_{22}, m_{32}, m_{42}]$$

while the signature $[m]$ single in the whole $W^{p,q}$ (i.e., the same for all $V_k^{p,q}$'s) is identified with the first row,

$$[m] \equiv [m_{13}, m_{23}, m_{33}, m_{43}].$$

The actions of the generators E_{ij} on the basis (22) are given in [24] or can be calculated by using the method explained there. The basis vector (22) with $m_{11} = m_{12}$ and $m_{31} = m_{32}$

$$(M)_k = \begin{bmatrix} m_{13} & m_{23} & m_{33} & m_{43} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{12} & 0 & m_{32} & 0 \end{bmatrix}_k \qquad (23)$$

annihilated by E_{12} and E_{34} is, by definition, the highest weight vector of the submodule $V_k^{p,q}([m]_k)$. For $k = 0$ the highest weight vector of the submodule $V_0^{p,q}([m])$

$$(M)_0 \equiv (M) = \begin{bmatrix} m_{13} & m_{23} & m_{33} & m_{43} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{13} & 0 & m_{33} & 0 \end{bmatrix} \qquad (24)$$

is, in addition, also annihilated by the odd generator E_{23} and, therefore, simultaneously represents the highest weight vector of both $V_0^{p,q}([m])$ and $W^{p,q}([m])$. A monomial of the form

$$|\theta_1, \theta_2, \theta_3, \theta_4\rangle := (E_{41})^{\theta_1}(E_{31})^{\theta_2}(E_{42})^{\theta_3}(E_{32})^{\theta_4} \qquad \theta_i = 0, 1 \qquad (25)$$

would shift a subspace $V_k^{p,q}$ to another subspace $V_l^{p,q}$ with $l > k$. So here we would call the former a higher (weight) subspace with respect to the latter called a lower (weight) subspace.

Proposition: *The induced module $W^{p,q}[m]$ constructed is irreducible if and only if*

$$\left[h_2^0\right]\left[h_1^0 + h_2^0 + 1\right]\left\{-\frac{q}{p}\left[h_2^0 - 1\right]\left[h_3^0 + 1\right] + \left[h_2^0\right]\left[h_3^0\right]\right\}\left\{-q^{-h_2^0+1}p^{-h_3^0-1}\left[h_1^0 + 1\right]\right.$$
$$- q^{h_1^0}\left(\frac{q}{p}\right)^{-h_2^0+1}\left[h_2^0 - 1\right]\left[h_3^0 + 1\right] + q^{h_1^0}\left(\frac{q}{p}\right)^{-h_2^0}\left[h_2^0\right]\left[h_3^0\right]$$
$$\left. + \frac{q}{p}\left(-q^{-h_2^0} + q^{-h_2^0-2}\right)\left[h_3^0\right]\left(q^{h_1^0+1} + \frac{q^2}{p^2}\left[h_1^0\right]\right)\right\} \neq 0 \qquad (26)$$

where $h_1^0 = m_{13} - m_{23}, h_2^0 = m_{23} + m_{33}, h_3^0 = m_{33} - m_{43}$.

The irreducible module $W^{p,q}$ constructed with keeping the condition (26) valid is called typical, otherwise, we say it is an indecomposable module. In the latter case, however, there always exists a maximal invariant submodule $I_h^{p,q}$ (of class $h, h = 1, 2, \ldots$) of $W^{p,q}$ and the compliment to $I_h^{p,q}$ subspace of $W^{p,q}$ is not invariant under $U_{p,q}[gl(2/2)]$ transformations. The representation carried in the factor module $W^{p,q}/I_h^{p,q}$ is irreducible and called a non-typical representation of $U_{p,q}[gl(2/2)]$. It can be shown that these typical and non-typical representations contain all classes of finite-dimensional irreducible representations of $U_{p,q}[gl(2/2)]$.

As every subspace $V_k^{p,q}, k = 0, 1, \ldots, 15$, is close and already irreducible under the even subalgebra $U_{p,q}[gl(2/2)_0]$, to see if $W^{p,q}$ is an irreducible module of $U_{p,q}$ it remains to consider the action of its odd generators only. By construction (see equations (17)–(21)) the module $W^{p,q}$ is at least indecomposable since any its subspace $V_k^{p,q}$, $1 \leqslant k \leqslant 15$, including the lowest one $V_{15}^{p,q}$, can be always reached from higher subspaces $V_l^{p,q}$, $0 \leqslant l < k$, including the highest one $V_0^{p,q}$, acted by the monomials $|\theta_1, \theta_2, \theta_3, \theta_4\rangle$ given in (25). Contrarily, the monomials

$$\langle\theta_1, \theta_2, \theta_3, \theta_4| := (E_{14})^{\theta_1}(E_{13})^{\theta_2}(E_{24})^{\theta_3}(E_{23})^{\theta_4} \tag{27}$$

send us to the opposite direction: from lower subspaces to higher ones. Thus, the module $W^{p,q}$ is irreducible if and only if $V_0^{p,q}$ is reachable from the lowest subspace $V_{15}^{p,q}$ under the action of the operators (27). The most optimal way to see that is to act on a vector of the subspace $V_{15}^{p,q}$ by the monomial $E_{14}E_{13}E_{24}E_{23}$, i.e., the monomial (27) with all θ_i's $= 1$ but not less (an action of a shorter monomial on $V_{15}^{p,q}$ should not reach $V_0^{p,q}$). Since $V_{15}^{p,q}$ is an irreducible module of $U_{p,q}[gl(2/2)_0]$, it is simplest but enough to consider when the highest weight vector $E_{41}E_{31}E_{42}E_{32}(M)$ of $V_{15}^{p,q}$ under the action of $E_{14}E_{13}E_{24}E_{23}$ reaches (or we can say, returns to) $V_0^{p,q}$. In other words, the module $W^{p,q}$ is irreducible if and only if the condition

$$E_{23}E_{24}E_{13}E_{14}E_{41}E_{31}E_{42}E_{32}(M) \neq 0 \tag{28}$$

holds. This condition in turn can be proved (for $p, q \neq 0$) to be equivalent to the condition

$$\begin{aligned}
[H_2][H_1 + H_2 + 1] \left\{ -\frac{q}{p}[H_2 - 1][H_3 + 1] + [H_2][H_3] \right\} \Bigg\{ -q^{-H_2+1} p^{-H_3-1}[H_1 + 1] \\
- q^{H_1} \left(\frac{q}{p}\right)^{-H_2+1} [H_2 - 1][H_3 + 1] + q^{H_1} \left(\frac{q}{p}\right)^{-H_2} [H_2][H_3] \\
+ \frac{q}{p} \left(-q^{-H_2} + q^{-H_2-2}\right) [H_3] \left(q^{H_1+1} + \frac{q^2}{p^2}[H_1] \right) \Bigg\} (M) \neq 0
\end{aligned} \tag{29}$$

which is nothing but (26) with h_i^0 being eigenvalues of H_i on the highest weight vector (M). The proposition is, thus, proved.

4. Conclusion

The two-parametric quantum superalgebra $U_{p,q}[gl(2/2)]$ was introduced in [24, 29]. Its representations constructed by the method described in [24] are either irreducible (when the condition (26) is kept) or indecomposable (when the condition (26) is violated). The irreducible representations in the former case are called typical. In the case of indecomposable representations, however, irreducible representations can always be extracted. One such irreducible representation called non-typical is simply a factor representation in a factor subspace of the original indecomposable module factorized by its maximal invariant subspace. All the typical and non-typical representations are constructed in such a way that they contain all classes of finite-dimensional irreducible representations of $U_{p,q}[gl(2/2)]$. In conclusion, let us emphasize that the condition (26) and the representations become more interesting at roots of unity but they, even at generic deformation parameters, are non-trivial deformations from the classical analogues [33] in the sense that the former cannot be found from the latter by replacing in appropriate places the ordinary brackets with the quantum deformation ones, unlike many one-parametric cases. We hope that the present results (and also previous ones [20–24, 33]) are physically useful and can be applied to investigating different physics models such as conformal field theory models [30–32] and solvable models of correlated electrons (see for example [34–36] and references therein).

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